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Finite N matrix models of noncommutative gauge theory

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Abstract: We describe a unitary matrix model which is constructed from discrete analogs of the usual projective modules over the noncommutative torus and use it to construct a lattice version of noncommutative gauge theory. The model is a discretization of the noncommutative gauge theories that arise from toroidal compactification of Matrix theory and it includes a recent proposal for a non-perturbative definition of noncommutative Yang-Mills theory in terms of twisted reduced models. The model is interpreted as a manifestly star-gauge invariant lattice formulation of noncommutative gauge theory, which reduces to ordinary Wilson lattice gauge theory for particular choices of parameters. It possesses a continuum limit which maintains both finite spacetime volume and finite noncommutativity scale. We show how the matrix model may be used for studying the properties of noncommutative gauge theory.

Keywords: Lattice Gauge Field Theories, M(atrix) Theories.

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1. Introduction

Noncommutative gauge theory first appeared in string theory within the framework of toroidal compactifications of Matrix theory [1]. It was argued that compactification on a noncommutative torus corresponds in 11-dimensional supergravity to null tori with a non-vanishing light-like component of the background three-form tensor field. Subsequently it was realized that the deformation to a noncommutative space can be described in type-II string theory as the effect of turning on a constant Neveu-Schwarz two-form tensor field $B_{\mu\nu}$ in the worldvolumes of D-branes [2, 3]. The parameter θ which deforms the space of functions on the worldvolume to a noncommutative algebra is related to the B-field background by $\theta \sim B^{-1}$. The low-energy effective field theory for the gauge fields living on the D-brane worldvolume can be described by a noncommutative gauge theory.

A non-trivial issue concerns the renormalizability of such gauge theories, given their unusual non-polynomial interactions. The perturbative renormalization properties of noncommutative Yang-Mills theory have been studied in [4]. In this paper we will present a constructive definition of noncommutative Yang-Mills theory which is the analog of the usual Wilson lattice gauge theory [5] in the commutative case. Such a model has the potential of clarifying issues of renormalization as well as shedding light on non-perturbative aspects of the gauge theory. A concrete definition of noncommutative gauge theory has been proposed recently in [6], and further studied in [7, 8], based on a large N reduced model [9]–[11]. In this case an ultraviolet regularization is naturally introduced at finite N and is removed in the large N limit with an appropriate fine-tuning of the gauge coupling constant. One expects the resulting theory in the continuum limit to have three scale parameters, the extent L of the space-time, the scale λ of noncommutativity, and the usual gauge theoretic scale parameter Λ. However, it is found that L/λ scales as $\sqrt{N}$, which means that one is inevitably led either to a finite L with $\lambda = 0$ (commutative finite space) or to a finite λ with $L = \infty$ (noncommutative infinite space) [7]. In the following we will show that there exists a more general constructive definition of noncommutative gauge theory which possesses a continuum limit whereby noncommutativity is compatible with a finite volume space.

The noncommutative gauge theory that naturally arises from toroidal compactification of Matrix Theory [1] comes from the matrix model which is obtained by dimensionally reducing ordinary Yang-Mills theory to a point [11]. The action is

$$\mathcal{S} = -\frac{1}{g^2}\sum_{\mu\neq\nu} \mathrm{tr}\,[X_\mu, X_\nu]^2 \,, \tag{1.1}$$

where X_μ, $\mu = 1, \dots, d$, are $N \times N$ hermitean matrices and d is the dimension of spacetime. To describe the compactification of this model on, say, a two-torus of radii R_1 and R_2, one needs to restrict the action (1.1) to those matrices X_μ that remain in the same gauge orbit after a shift by $2\pi R_\mu\,\mathbf{1}_N$ in the direction μ. This is tantamount to finding configurations for which there exists unitary matrices Ω_μ, $\mu = 1, 2$, which generate the quotient conditions [1, 12, 13]

$$X_\mu + 2\pi R_\mu \delta_{\mu\nu}\,\mathbf{1}_N = \Omega_\nu X_\mu \Omega_\nu^\dagger \,. \tag{1.2}$$

Taking the trace of both sides of this condition shows that these equations cannot be solved by finite-dimensional matrices. It is, however, straightforward to solve them by self-adjoint operators on an infinite-dimensional Hilbert space. The basic observation [1] is that consistency of the conditions (1.2) when represented on this Hilbert space requires that

$$\Omega_1\Omega_2 = \mathrm{e}^{2\pi i\theta}\,\Omega_2\Omega_1 \tag{1.3}$$

for some real number θ. This means that the operators Ω_μ generate the noncommutative two-torus with noncommutativity parameter θ.

The operators Ω_μ may be represented on a Hilbert space $\mathcal{H}_{p,q}$ of functions $f_k(s)$ where $s \in \mathbb{R}$ and $k \in \mathbb{Z}_q$ [1]. By introducing a fixed, fiducial derivation ∇_μ on this Hilbert space which satisfies

$$[\nabla_\mu, \Omega_\nu] = 2\pi i\,\delta_{\mu\nu}\,\Omega_\nu \,, \tag{1.4}$$

a generic solution of (1.2) may be taken to be the sum of ∇_μ and a fluctuating part,

$$X_\mu = iR_\nu\delta_{\mu\nu}\nabla_\nu + A_\mu(Z) \,, \tag{1.5}$$

where Z_μ generate the algebra of operators which commute with the Ω_μ's. There is a standard construction of these operators on the Hilbert space $\mathcal{H}_{p,q}$ [1]. In noncommutative geometry this simply corresponds to the algebraic construction of vector bundles over the noncommutative torus and the solutions (1.5) are just connections on these bundles [14]. The bundle $\mathcal{H}_{p,q}$ is characterized by its "commutative" rank $p = \dim \mathcal{H}_{p,q}|_{\theta=0}$ and its magnetic flux q which is taken to be the constant curvature of the fixed connection ∇_μ, $2\pi q = \mathrm{Tr}\, i\,[\nabla_\mu, \nabla_\nu]$. The gauge fields $A_\mu(Z)$ are then functions on a dual noncommutative torus and the substitution of (1.5) back into the action (1.1) gives Yang-Mills theory on this dual noncommutative torus.

This construction has been reinterpreted recently in terms of open string quantization in the presence of a constant background B-field [3]. The modules $\mathcal{H}_{p,q}$ are constructed from the boundary worldsheet theory appropriate to one end of an open string terminating on a D2-brane and the other end on a configuration of p coincident D2-branes carrying q units of D0-brane charge. In this paper we will present a construction which is a straightforward discretization of the above formalism in terms of an $N \times N$ unitary matrix model. We shall recover all the parameters labeling the continuum theory in the large N limit. In particular, in this formulation finite λ is compatible with finite L. The proposal in [6] can be regarded as a special case, from which it becomes transparent why finite λ is not compatible with finite L in that instance. Thus the ensuing matrix model naturally interpolates between the model in [6] and the continuum formalism in [1] for the matrix model of M-theory [12]. We will show that our model can be interpreted as a manifestly star-gauge invariant lattice formulation of noncommutative gauge theory, which reduces to Wilson's lattice gauge theory [5] for particular choices of the parameters even at finite N. We shall also describe how various aspects of noncommutative gauge theory can be systematically studied within the matrix model formalism.

2. The unitary matrix model

We will describe the construction in the simplest two-dimensional case, but the generalization to arbitrary even dimension is straightforward. The model we consider is just a twisted Eguchi-Kawai model [9, 10], but with a certain constraint imposed on the matrices. The action is

$$S = -\beta \sum_{\mu \neq \nu} Z_{\mu\nu}\, \mathrm{tr}\left(U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger\right), \tag{2.1}$$

where U_μ $(\mu = 1, 2)$ are $N \times N$ unitary matrices and $Z_{\mu\nu} = Z_{\nu\mu}^*$ is a phase factor called the "twist". The constraint we impose on the matrices U_μ is

$$\Omega_\nu U_\mu \Omega_\nu^\dagger = \mathrm{e}^{2\pi i \delta_{\mu\nu} r_\mu / N}\, U_\mu\,, \tag{2.2}$$

where r_μ $(\mu = 1, 2)$ are integers which we will specify below. The constraint (2.2) is the exponentiation of eq. (1.2). Now, however, the only condition required is that the U_μ's are traceless unitary matrices. It therefore represents a finite dimensional version of the quotient conditions for Matrix theory.

To solve the consistency conditions (1.3), we take the unitary matrices

$$\Omega_1 = (\Gamma_2)^m \otimes (\tilde{\Gamma}_1)^{\dagger p}\,, \qquad \Omega_2 = (\Gamma_1)^m \otimes (\tilde{\Gamma}_2)^{\dagger}\,, \tag{2.3}$$

where Γ_μ and $\tilde{\Gamma}_\mu$ are unitary matrices of dimension M and q, respectively, which satisfy the Weyl-'t Hooft commutation relations

$$\Gamma_1\Gamma_2 = \mathrm{e}^{2\pi i/M}\,\Gamma_2\Gamma_1\,, \qquad \tilde{\Gamma}_1\tilde{\Gamma}_2 = \mathrm{e}^{2\pi i/q}\,\tilde{\Gamma}_2\tilde{\Gamma}_1\,. \tag{2.4}$$

These algebras can be represented by the shift and clock matrices $(\Gamma_1)_{jk} = \delta_{j+1,k}$, $(\Gamma_2)_{jk} = (\mathrm{e}^{2\pi i/M})^{j-1}\,\delta_{jk}$, and similarly for the $\tilde{\Gamma}_\mu$. The integers M and q satisfy $N = Mq$ and we take $M = mnq$. The deformation parameter θ of eq. (1.3) is given by

$$\theta = \frac{p}{q} - \frac{m}{nq}\,. \tag{2.5}$$

The incorporation of two independent integers m and n in the above will enable us to take a large-N limit whereby the appropriate continuum limit is reproduced. They play a certain dual role to one another as we shall see. We will also see later on that the meaning of the integers p and q will be the same as in the description of the modules $\mathcal{H}_{p,q}$ above.

Our first task is to solve the constraint (2.2). We take the simplest particular solution $U_\mu = D_\mu$ associated with[1] $r_1 = r_2 = mq$:

$$D_1 = (\Gamma_1)^\dagger \otimes \mathbb{1}_q\,, \qquad D_2 = \Gamma_2 \otimes \mathbb{1}_q\,. \tag{2.6}$$

These operators will become fixed covariant derivatives in the continuum limit. We then decompose U_μ using D_μ as

$$U_\mu = \tilde{U}_\mu D_\mu\,, \tag{2.7}$$

where $\tilde{U}_\mu$ are unitary matrices which satisfy the constraint

$$\Omega_\nu \tilde{U}_\mu \Omega_\nu^\dagger = \tilde{U}_\mu\,. \tag{2.8}$$

These constrained matrices will become the gauge fields of the model in the continuum limit and they can be constructed as follows. Assuming that p and q are co-prime, we choose integers a and b such that

$$ap + bq = 1\,. \tag{2.9}$$

We then introduce unitary matrices

$$Z_1 = (\Gamma_2)^n \otimes (\tilde{\Gamma}_1)^\dagger\,, \qquad Z_2 = (\Gamma_1)^{\dagger n} \otimes (\tilde{\Gamma}_2)^a\,, \tag{2.10}$$

[1] One can consider a more general particular solution $U_\mu = (D_\mu)^{l_\mu}$ where l_μ are integers. If l_μ is a divisor of mq, then this solution will reproduce in the continuum limit the noncommutative gauge theory associated with a torus of modulus $R_1/R_2 = l_1/l_2$.

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which commute with Ω_μ. The commutation relation of the Z_μ is

$$Z_1 Z_2 = \mathrm{e}^{2\pi i\theta'} Z_2 Z_1 \,, \tag{2.11}$$

where θ' is given by

$$\theta' = \frac{n}{mq} - \frac{a}{q} \tag{2.12}$$

and it is related to θ through the discrete Möbius transformation

$$\theta' = \frac{a\theta + b}{p - q\theta} \,. \tag{2.13}$$

In the continuum, the transformation law (2.13) would be just that between Morita equivalent noncommutative tori [14]. In fact, by identifying θ with a constant Neveu-Schwarz two-form field, it is just the T-duality transformation rule for the B-field [3, 15]. The relationship between Morita equivalence and duality [15] means that certain noncommutative gauge theories are physically equivalent to one another. We will return to this point later on.

Using Z_μ, we can define a basis for the solution space of (2.8) as

$$J_{m_1,m_2} = (Z_2)^{m_1} (Z_1)^{m_2} \, \mathrm{e}^{\pi i \theta' m_1 m_2} \,, \tag{2.14}$$

where the phase factor is included so that

$$J_{-m_1,-m_2} = (J_{m_1,m_2})^\dagger \,. \tag{2.15}$$

Since $(Z_\mu)^{mq} = \mathbb{1}_N$, J_{m_1,m_2} is quasi-periodic with respect to m_1 and m_2 with period mq. We can therefore restrict the integers m_1 and m_2 to run from $-\frac{mq-1}{2}$ to $\frac{mq-1}{2}$ (this requires mq odd). It will prove convenient to introduce a lattice with $(mq)^2$ sites on the torus and to instead work with the basis defined by

$$\Delta(x) = \sum_{m_1,m_2} J_{m_1,m_2} \, \mathrm{e}^{-2\pi i \epsilon_{\mu\nu} m_\mu x_\nu / L} \,, \tag{2.16}$$

where $x_\mu = 0, \epsilon, \ldots, \epsilon(mq-1)$ belongs to the lattice of the spacing ϵ and the extent of the lattice is

$$L = \epsilon mq. \tag{2.17}$$

We have defined $\Delta(x)$ in such a way that the identities

$$\begin{aligned} \frac{1}{(mq)^2} \sum_x \Delta(x) &= \mathbb{1}_N \,, \\ D_\mu \Delta(x) D_\mu^\dagger &= \Delta(x - \epsilon\hat{\mu}) \end{aligned} \tag{2.18}$$

hold. Here $\epsilon\hat{\mu}$ denotes a shift by ϵ of x_μ only. The relation (2.18) expresses the identification of the matrices D_μ as discrete covariant derivatives. Note also that

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$\Delta(x)$ is hermitean, due to (2.15), and is periodic with respect to x_1 and x_2 with period L. The proof of completeness the generators (2.16) is given in appendix A. Given this complete set of solutions, we can write a general solution to (2.8) as

$$\tilde{U}_\mu = \frac{1}{(mq)^2} \sum_x \mathcal{U}_\mu(x) \Delta(x) \,. \tag{2.19}$$

Using orthogonality we can invert (2.19) to give (see appendix A)

$$\mathcal{U}_\mu(x) = \frac{1}{N} \operatorname{tr}(\tilde{U}_\mu \, \Delta(x)) \,. \tag{2.20}$$

In order that the right-hand side of eq. (2.19) is a unitary matrix, the coefficients $\mathcal{U}_\mu(x)$ should satisfy a certain condition which will be given below.

Having solved the constraint, our next task is to rewrite our model entirely in terms of $\mathcal{U}_\mu(x)$, which are the gauge fields of the theory. For this, we use the identity (2.20) to regard $\Delta(x)$ as a map from the space of $N \times N$ matrices which commute with Ω_μ to the space of fields on a periodic $L \times L$ lattice. We use the natural definition of a product of two lattice fields $f_1(x)$ and $f_2(x)$:

$$f_1(x) \star f_2(x) \stackrel{\text{def}}{=} \frac{1}{N} \operatorname{tr}(f_1 f_2 \, \Delta(x)) \,, \tag{2.21}$$

where f_i are the $N \times N$ matrices defined by $f_i = (mq)^{-2} \sum_x f_i(x) \Delta(x)$. This product is associative but not commutative. One can write it explicitly in terms of $f_i(x)$ as

$$f_1(x) \star f_2(x) = \frac{1}{(mq)^2} \sum_{y,z} f_1(y) f_2(z) \, \mathrm{e}^{2iB\epsilon_{\mu\nu}(x_\mu - y_\mu)(x_\nu - z_\nu)} \,, \tag{2.22}$$

where

$$B = \frac{2\pi}{\theta' L^2} \,. \tag{2.23}$$

These formulas are similar to ref. [8]. (See also [16] for earlier works in this regard.) The product (2.22) can be considered as the lattice version of the star product in noncommutative geometry. To see this, we note that in the continuum the star product of two functions $f_1(x)$ and $f_2(x)$ may be defined as

$$f_1(x) \star f_2(x) \stackrel{\text{def}}{=} f_1(x) \exp\left(i\frac{1}{2} \overleftarrow{\partial_\mu} \, \theta_{\mu\nu} \overrightarrow{\partial_\nu} \right) f_2(x) \,. \tag{2.24}$$

Using a Fourier transformation, this definition can be turned into an integral form

$$f_1(x) \star f_2(x) = \iint \mathrm{d}y \; \mathrm{d}z \; K(x-y, x-z) \, f_1(y) f_2(z) \,, \tag{2.25}$$

where the integration kernel K is given by

$$K(x-y, x-z) = \frac{1}{\pi^d |\det \theta_{\mu\nu}|} \mathrm{e}^{-2i(\theta^{-1})_{\mu\nu}(x_\mu - y_\mu)(x_\nu - z_\nu)} \,. \tag{2.26}$$

The expression (2.22) can be obtained from (2.25) just by restricting the variables x, y, z to run over lattice points. In this sense, the product (2.22) is a natural lattice counterpart of the star product in the continuum. We shall therefore call (2.22) a star product in what follows.

Using the star product, we can write down the condition on $\mathcal{U}_\mu(x)$ which is required for $\tilde{U}_\mu$ to be unitary as

$$\mathcal{U}_\mu(x)^* \star \mathcal{U}_\mu(x) = 1\,. \tag{2.27}$$

In other words, the lattice fields $\mathcal{U}_\mu(x)$ must be star-unitary. We may now rewrite the action (2.1) as

$$\begin{aligned}
S &= -\beta \frac{1}{(mq)^2} \sum_x \sum_{\mu\neq\nu} Z_{\mu\nu}\, \mathrm{tr}\left[U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger \Delta(x)\right] \\
&= -\beta \frac{1}{(mq)^2} \sum_x \sum_{\mu\neq\nu} Z_{\mu\nu}\, \mathrm{tr}\left[\tilde{U}_\mu D_\mu \tilde{U}_\nu D_\nu D_\mu^\dagger \tilde{U}_\mu^\dagger D_\nu^\dagger \tilde{U}_\nu^\dagger \Delta(x)\right] \\
&= -\beta \frac{1}{(mq)^2} \sum_x \sum_{\mu\neq\nu} \tilde{Z}_{\mu\nu}\, \mathrm{tr}\left[\tilde{U}_\mu (D_\mu \tilde{U}_\nu D_\mu^\dagger)(D_\nu \tilde{U}_\mu^\dagger D_\nu^\dagger) \tilde{U}_\nu^\dagger \Delta(x)\right] \\
&= -\beta \frac{n}{m} \sum_x \sum_{\mu\neq\nu} \tilde{Z}_{\mu\nu}\, \mathcal{U}_\mu(x) \star \mathcal{U}_\nu(x+\epsilon\hat{\mu}) \star \mathcal{U}_\mu^*(x+\epsilon\hat{\nu}) \star \mathcal{U}_\nu^*(x)\,,
\end{aligned} \tag{2.28}$$

where $\tilde{Z}_{\mu\nu} = Z_{\mu\nu}\,\mathrm{e}^{-2\pi i\epsilon_{\mu\nu}/M}$ can be considered as a background rank-two tensor field. One can make $\tilde{Z}_{\mu\nu} = 1$ by choosing $Z_{\mu\nu} = \mathrm{e}^{2\pi i\epsilon_{\mu\nu}/M}$. Then the vacuum configuration is given by $\tilde{U}_\mu = \mathbb{1}_N$, which corresponds to $\mathcal{U}_\mu(x) = 1$, up to the symmetry of the model which we now proceed to discuss.

The action (2.1) and the Haar integration measure for the matrices U_μ are invariant under the $\mathrm{SU}(N)$ transformations

$$U_\mu \to g\, U_\mu\, g^\dagger\,. \tag{2.29}$$

The constraint (2.2) in general breaks this symmetry down to a subgroup of $\mathrm{SU}(N)$. However, the constrained model is still invariant under (2.29) for any g that commutes with Ω_μ. We can represent such a g in terms of a function $g(x)$ on the lattice as

$$g = \frac{1}{(mq)^2} \sum_x g(x)\Delta(x)\,, \tag{2.30}$$

where $g(x)$ should satisfy $g(x)^* \star g(x) = 1$, but is otherwise arbitrary. The transformation (2.29) can now be interpreted in terms of $\mathcal{U}_\mu(x)$ as

$$\mathcal{U}_\mu(x) \to g(x) \star \mathcal{U}_\mu(x) \star g^*(x+\epsilon\hat{\mu})\,. \tag{2.31}$$

Therefore, the resulting theory of the lattice field $\mathcal{U}_\mu(x)$ is manifestly invariant under this star-gauge symmetry.

We can show that the theory (2.28) reduces to Wilson's lattice gauge theory [5] for particular choices of the parameters. Note that we can always make $\theta' = 0$ by taking $n = ma$. In this case, the star product becomes the ordinary product of functions. Therefore, $\mathcal{U}_\mu(x)$ becomes an ordinary U(1) field on the lattice and the action (2.28) becomes the ordinary Wilson plaquette action. We can also show that the integration measure for $\mathcal{U}_\mu(x)$ is actually the Haar measure for integration over the group $\mathrm{U}(1)^{2(mq)^2}$. Note that the Haar measure for the $N \times N$ matrices U_μ and the constraint (2.2) are invariant under $U_\mu \to gU_\mu$ for any g which commutes with Ω_μ. This can be translated into the invariance of the integration measure for $\mathcal{U}_\mu(x)$ under $\mathcal{U}_\mu(x) \to g(x)\mathcal{U}_\mu(x)$. The uniqueness of a measure with such an invariance proves our statement. Thus, our lattice formulation of noncommutative gauge theory includes Wilson's lattice gauge theory on a periodic lattice of finite extent as the $\theta' = 0$ case, even at finite N. We remark that in this case, although the Z_μ matrices can be taken to be diagonal, the $(mq)^2$ degrees of freedom of the lattice gauge theory are contained in the $N = mnq^2 = a(mq)^2$ diagonal elements of $\tilde{U}_\mu$.

Going back to the general case of arbitrary θ', let us now consider the continuum limit of the model (2.28) when the lattice spacing $\epsilon \to 0$. We introduce the continuum field $\tilde{A}_\mu$ and operator d_μ through

$$\tilde{U}_\mu = \mathrm{e}^{i\epsilon\tilde{A}_\mu}\,, \qquad D_\mu = \mathrm{e}^{i\epsilon d_\mu}\,. \tag{2.32}$$

The large N limit dictated by the continuum theory [1] is $m \sim n \sim \sqrt{N}$ and $\epsilon \sim 1/\sqrt{N}$ with fixed a, b, p and q. Both L given by (2.17) and B given by (2.23) are finite in such a large N limit. The resulting gauge theory is constructed from connections of a rank p bundle of magnetic flux q. We will see this explicitly in the next section. Note that the field theory (2.28) is actually of rank 1. This is one of the characteristic features of Morita equivalence or alternatively of T-duality transformations between different brane worldvolume field theories. The original SU(p) Yang-Mills theory on the noncommutative torus with deformation parameter θ is physically equivalent to a U(1) Yang-Mills theory on a dual torus with noncommutativity parameter (2.13) that implicitly contains the information about the rank p of the underlying vector bundle. The case $q = 0$, representing a trivial gauge bundle, can also be constructed and will be presented elsewhere.

However, as far as the continuum limit of the lattice theory is concerned, we need only send m to infinity, but not necessarily n. If n is finite as $m \to \infty$, this does not lead to the solutions constructed in the continuum [1] for hermitean operators and is instead associated with unitary operators acting on periodic functions of $0 \le s < nq$. The particular case of $q = n = 1$, for which the condition (2.2) is trivial and our model reduces to the ordinary, unconstrained twisted Eguchi-Kawai model, is of this type. It corresponds to the interpretation of the twisted large N reduced model in terms of noncommutative gauge theory which was proposed in [6]. Since $\theta' = 1/N$ in that case, in order to make B finite one needs $\epsilon \sim 1/\sqrt{N}$, which inevitably makes the physical extent of the torus scale as $L \sim \sqrt{N}$, reproducing the observation made in [7]. Note that the issue of whether or not a continuum limit really exists is a dynamical question that can be addressed, for example, by Monte Carlo simulation. A numerical simulation of the two-dimensional Eguchi-Kawai model has been done in [17], where a non-trivial large N scaling behavior was found with the parameter $N\epsilon^2$ fixed, which is exactly the large N limit required to make the physical scale of noncommutativity finite. This in itself means that noncommutative gauge theory with a background tensor field can be constructively defined.

3. Observables of noncommutative gauge theory

We will now describe how the properties of noncommutative gauge theory can be completely reformulated in the language of the unitary matrix model above. Let us define a lattice path which consists of s links by $C = \{\hat{\mu}_1, \ldots, \hat{\mu}_s\}$ and $C^{-1} = \{\hat{\mu}_s, \ldots, \hat{\mu}_1\}$ for an opposite orientation. The path C connects lattice sites separated by the vector $\ell^\mu = \xi_s^\mu$ while

$$\xi_i^\mu = \epsilon \sum_{j=1}^{i} \hat{\mu}_j \tag{3.1}$$

belongs to C. We introduce the following products of matrices along the path:

$$\begin{aligned} D(C) &= \prod_{j=1}^{s} D_{\mu_j}\,, \qquad D(C^{-1}) = D(C)^\dagger\,; \\ U(C) &= \prod_{j=1}^{s} \left(\tilde{U}_{\mu_j} D_{\mu_j}\right)\,. \end{aligned} \tag{3.2}$$

Given the property (2.18) we then have

$$\Delta(x+\ell) = D(C)\Delta(x)D(C^{-1})\,, \tag{3.3}$$

where the right-hand side is path-independent because of the properties of the D_μ. This results in the formula

$$\frac{1}{N}\,\mathrm{tr}(A\,\Delta(x)) \star \frac{1}{N}\,\mathrm{tr}(B\,\Delta(x+\ell)) = \frac{1}{N}\,\mathrm{tr}\left(AD(C)BD(C^{-1})\,\Delta(x)\right)\,, \tag{3.4}$$

provided that A and B belong to the commutant of the algebra generated by Ω_μ. Using (3.4), we can construct the matrix analog of the noncommutative phase factor along the lattice path which defines parallel transport for the gauge bundle in the continuum limit,

$$\mathcal{U}(x;C) \stackrel{\mathrm{def}}{=} \star\prod_{j=1}^{s} \mathcal{U}_{\mu_j}(x+\xi_{j-1}) = \frac{1}{N}\,\mathrm{tr}\left(U(C)D(C^{-1})\,\Delta(x)\right)\,, \tag{3.5}$$

where the product in the middle is the star product. Under the $\mathrm{SU}(N)$ gauge transformation (2.29) where $U(C) \to gU(C)g^\dagger$, the right-hand side of eq. (3.5) transforms as

$$\mathcal{U}(x;C) \to \frac{1}{N}\,\mathrm{tr}\,\left(gU(C)g^\dagger D(C^{-1})\,\Delta(x)\right) = g(x) \star \mathcal{U}(x;C) \star g^*(x+\ell) \tag{3.6}$$

as it should for the phase factor. This formula extends (2.31) to an arbitrary open path.

The continuum limit of the above construction is given by the large-N limit of the matrix model. We introduce $\mathrm{d}\xi^\mu = \epsilon\hat{\mu}$, so that eq. (3.1) takes the form

$$\xi^\mu = \int \mathrm{d}\xi^\mu\,, \tag{3.7}$$

and write down the continuum analogs of eqs. (3.2) and (3.5) using (2.32) as

$$\begin{aligned} D(C) &= \mathrm{P}\,\exp\left(i\int_0^\ell \mathrm{d}\xi^\mu\; d_\mu\right), \\ U(C) &= \mathrm{P}\,\exp\left(i\int_0^\ell \mathrm{d}\xi^\mu\; (\tilde{A}_\mu + d_\mu)\right) \end{aligned} \tag{3.8}$$

and

$$\mathcal{U}(x;C) = \star\prod_\xi \left(1 + i\,\mathrm{d}\xi^\mu\,\mathcal{A}_\mu(x+\xi)\right). \tag{3.9}$$

Here we have defined the field $\mathcal{A}(x)$ by

$$\mathcal{U}_\mu(x) = \star\,\mathrm{e}^{i\epsilon\mathcal{A}_\mu(x)} \tag{3.10}$$

where the exponential is understood as a power series expansion with the star-product. Notice that the d_μ in eq. (3.8) cannot be absorbed by a shift of $\tilde{A}_\mu$ since d_μ does not commute with the Ω's. This is the difference between the present model and the continuum limit of the twisted Eguchi-Kawai model where this translation is usually done.

The phase factors (3.9) can be used to define a new class of observables in the matrix model, associated with noncommutative gauge theory. The standard closed Wilson loops $W(C)$ of twisted reduced models [10] which are invariant under (2.29) can be expressed via $\mathcal{U}(x;C)$ as

$$\begin{aligned} W(C) &\equiv \frac{1}{N}\,\mathrm{tr}\,D(C^{-1})\,\frac{1}{N}\,\mathrm{tr}\,U(C) \\ &= \frac{1}{(mq)^2}\sum_x \frac{1}{N}\,\mathrm{tr}\left(U(C)D(C^{-1})\,\Delta(x)\right) = \frac{1}{(mq)^2}\sum_x \mathcal{U}(x;C)\,, \end{aligned} \tag{3.11}$$

since $D(C^{-1})$ is a c-number. Therefore, the analog of $W(C)$ in noncommutative gauge theory is a sum over lattice points of $\mathcal{U}(x;C)$, which is understood as the sum over translations of the closed path that preserve its shape. This object is star-gauge invariant due to this additional summation. For the simplest closed loop, i.e. the plaquette, it is used in constructing the action (2.1). What is rather surprising in noncommutative gauge theory is that one can actually construct a star-gauge invariant observable associated with an open path, as has been found in [7]. We will now describe how such observables appear in our model and point out an interesting consequence of the finiteness of the spacetime extent.

Star-gauge invariant quantities can be constructed out of (3.5) with the aid of a lattice function $S_\ell(x)$ which has the property

$$S_\ell(x) \star g(x) \star S_\ell(x)^{-1} = g(x+\ell) \tag{3.12}$$

for arbitrary star-unitary functions $g(x)$. Here again ℓ_μ is the relative separation vector between the two ends of the open loop. Star-gauge invariant quantities can then be defined by $(mq)^{-2}\sum_x S_\ell(x) \star \mathcal{U}(x;C)$. The property (3.12) in the matrix model becomes

$$S_\ell \Delta(x) S_\ell^{-1} = \Delta(x-\ell)\,, \tag{3.13}$$

where

$$S_\ell = \frac{1}{(mq)^2}\sum_x S_\ell(x)\Delta(x) \tag{3.14}$$

belongs to the commutant of the algebra generated by Ω_μ. Using the definition (2.16) and with a little algebra, we obtain that $S_\ell(x)$ should satisfy

$$S_\ell(x+\theta' L\hat{\mu}) = \mathrm{e}^{-2\pi i \epsilon_{\mu\nu}\ell_\nu/L} S_\ell(x)\,, \tag{3.15}$$

where L is given by (2.17). Assuming that $\theta' mq = n - ma$ and mq are co-prime, the only solution to (3.15) is

$$S_\ell(x) = \mathrm{e}^{2\pi i k\cdot x/L}\,, \tag{3.16}$$

where $k_\mu = 0, 1, \ldots, (mq-1)$ and

$$\ell_\mu = \theta' L \epsilon_{\mu\nu} k_\nu + n_\mu L\,, \tag{3.17}$$

with an integer vector n_μ. As is seen from (3.16), the ratio $2\pi k_\mu/L$ plays the role of the momentum variable and it is related to the distance vector ℓ_μ by eq. (3.17). The longer the open loop is, the larger the momentum $2\pi k_\mu/L$ should be. The discretization of momentum due to the finite extent of the torus leads us to an interesting consequence that ℓ_μ should also be discrete. In the commutative case when $\theta' = 0$, we obtain $\ell_\mu = n_\mu L$ reproducing the known fact that the only such gauge invariant quantities are the Polyakov loops (holonomies of noncontractable loops on the torus). It is remarkable that in noncommutative gauge theory on a finite volume there exist other objects of this kind with discretized values of the distance ℓ_μ. It remains discrete in the continuum limit since L is finite. This is the

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difference from the analogous quantities constructed in [7] for the IIB model, where ℓ_μ can be an arbitrary vector in the large N limit. The matrix description of the star-gauge invariant open loop is given by

$$\begin{aligned}\frac{1}{(mq)^2}\sum_x S_\ell(x)\star\mathcal{U}(x;C) &= \frac{1}{(mq)^2}\sum_x \frac{1}{N}\,\mathrm{tr}\left(U(C)D(C^{-1})S_\ell\,\Delta(x)\right)\\ &= \frac{1}{N}\,\mathrm{tr}\left(U(C)D(C^{-1})S_\ell\right),\end{aligned}\tag{3.18}$$

where $S_\ell = J_{k_2,-k_1}$ is given by (2.14) for the solution (3.16). Its star-gauge invariance can be directly checked by noting that $D(C^{-1})S_\ell$ in (3.18) belongs for ℓ given by eq. (3.17) to the commutant of the algebra generated by Z_μ, i.e. commutes with g.

The matrix model determines the dynamics of noncommutative gauge theory. Let us demonstrate how the classical equation of motion emerges in the matrix language. For simplicity we take the continuum limit using the relation (2.32). The continuum action reads

$$S[\tilde{A}] = \mathrm{tr}\left(\left(\tilde{F}_{\mu\nu}-f_{\mu\nu}\right)^2\right) + \mathrm{tr}\left(\alpha_{\mu\nu}\left[\tilde{A}_\mu,\Omega_\nu\right]\right)\tag{3.19}$$

where

$$\tilde{F}_{\mu\nu} = id_\mu\tilde{A}_\nu - id_\nu\tilde{A}_\mu + i\left[\tilde{A}_\mu,\tilde{A}_\nu\right].\tag{3.20}$$

Here $f_{\mu\nu}$ is the constant curvature of the gauge bundle given by

$$-i\,[d_\mu,d_\nu] = f_{\mu\nu}\,\mathbb{1}_N\tag{3.21}$$

where in the two dimensional case

$$f_{\mu\nu} = \frac{2\pi q}{p-q\theta}\,R_1R_2\,\epsilon_{\mu\nu}.\tag{3.22}$$

In the construction of section 2, $R_1 = R_2 = 1/\epsilon nq$ are the radii of the two-torus (see footnote 1). Eq. (3.22) is the standard formula for the curvature of the module $\mathcal{H}_{p,q}$ [1]. It should be understood, however, as being multiplied by the identity operator $\mathbf{1}_{p,q}$ with $\mathrm{Tr}\,\mathbf{1}_{p,q} = p-q\theta$, so that the integral curvature of the bundle is $\mathrm{Tr}\,f_{\mu\nu}/(2\pi R_1R_2) = q$. In the present case this trace operation corresponds to multiplying the curvature (3.22) by the dimensionless area factor $\sqrt{(R_1R_2)(L_1L_2)} = m/n$ giving the volume of a unit cell in the "phase space" of the d_μ's. This is analogous to the derivation of the dimension of the Hilbert space $\mathcal{H}_{p,q}$ presented in [1]. The (infinite) hermitean matrices in (3.19) are unconstrained while the constraints are taken into account by the Lagrange multipliers $\alpha_{\mu\nu}$. The action (3.19) is of the type considered in [18], but now with the additional constraints imposed on $\tilde{A}$.

The variational derivative

$$\frac{\delta}{\delta\mathcal{A}_\mu(x)}\,\mathcal{A}_\nu(y) = \delta_{\mu\nu}\,\delta(x-y)\tag{3.23}$$

can be represented in the matrix language as follows. Given (2.19), (2.32) and (3.10), we have

$$\tilde{A}_\mu = \int \mathrm{d}x \; \mathcal{A}(x)\Delta(x)\,, \tag{3.24}$$

which implies

$$\frac{\delta}{\delta \mathcal{A}_\mu(x)} = \mathrm{tr}\left(\frac{\partial}{\partial \tilde{A}_\mu}\,\Delta(x)\right). \tag{3.25}$$

Equation (3.23) is now reproduced as

$$\mathrm{tr}\left(\frac{\partial}{\partial \tilde{A}_\mu}\,\Delta(x)\right)\,\frac{1}{N}\,\mathrm{tr}\left(\tilde{A}_\nu\,\Delta(y)\right) = \delta_{\mu\nu}\,\frac{1}{N}\,\mathrm{tr}\left(\Delta(x)\Delta(y)\right) = \delta_{\mu\nu}\,\delta(x-y) \tag{3.26}$$

as it should. We can treat the matrix elements of $\tilde{A}_\nu$ as independent because of the completeness of the generators of the commutant. Notice that there is an ordinary product of the traces in (3.26) rather than the star product. Acting by (3.25) on the action (3.19), we get

$$\begin{aligned}\frac{\delta}{\delta \mathcal{A}_\nu(x)}\,S &= \mathrm{tr}\left(\left[d_\mu + \tilde{A}_\mu, \tilde{F}_{\mu\nu} - f_{\mu\nu}\right]\,\Delta(x)\right) + \mathrm{tr}\left(\alpha_{\mu\nu}\,[\Delta(x),\Omega_\mu]\right)\\ &= \mathrm{tr}\left(\left[d_\mu + \tilde{A}_\mu, \tilde{F}_{\mu\nu}\right]\,\Delta(x)\right).\end{aligned} \tag{3.27}$$

Eq. (3.27) reproduces the noncommutative Maxwell equation.

The matrix representation (3.25) of the variational derivative is actually most useful for deriving the Schwinger-Dyson equations of the quantum noncommutative theory and, in particular, the loop equations. To illustrate the technique, let us first calculate how the variation $\delta/\delta\mathcal{A}_\mu(z)$ acts on the noncommutative phase factor $\mathcal{U}(x;C)$, which determines the contact term in the loop equation [19]. Using (3.25), we get

$$\begin{aligned}\frac{\delta}{\delta \mathcal{A}_\nu(z)}\,\mathcal{U}(x;C) &= \mathrm{tr}\left(\frac{\partial}{\partial \tilde{A}_\nu}\,\Delta(z)\right)\frac{1}{N}\,\mathrm{tr}\left(U(C)D(C^{-1})\,\Delta(x)\right)\\ &= i\int_0^\ell \mathrm{d}\xi^\nu\,\frac{1}{N}\,\mathrm{tr}\left(U(C_1)\,\Delta(z)U(C_2)\,D(C^{-1})\Delta(x)\right)\\ &= i\int_0^\ell \mathrm{d}\xi^\nu\,\mathcal{U}(x;C_1)\star\delta(x+\xi-z)\star\mathcal{U}(x+\xi;C_2)\,,\end{aligned} \tag{3.28}$$

where C_1 and C_2 are the parts of the contour C, $C=C_1C_2$, separated by ξ. We can similarly calculate how the area operator $\partial_\mu\delta/\delta\sigma^{\mu\nu}(z)$ $(z\in C)$ acts on $\mathcal{U}(x;C)$. This calculation is purely geometrical and gives

$$\begin{aligned}&\partial_\mu\,\frac{\delta}{\delta\sigma^{\mu\nu}(z)}\,\mathcal{U}(x;C) = \frac{1}{N}\,\mathrm{tr}\left(U(C_1)\left[d_\mu+\tilde{A}_\mu,\tilde{F}_{\mu\nu}\right]U(C_2)D(C^{-1})\,\Delta(x)\right)\\ &= -i\,\mathcal{U}(x;C_1)\star\left(\partial_\mu\mathcal{F}_{\mu\nu}+i\mathcal{A}_\mu\star\mathcal{F}_{\mu\nu}-i\mathcal{F}_{\mu\nu}\star\mathcal{A}_\mu\right)(z)\star\mathcal{U}(z;C_2)\,,\end{aligned} \tag{3.29}$$

where

$$\mathcal{F}_{\mu\nu} = \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu + i\mathcal{A}_\mu\star\mathcal{A}_\nu - i\mathcal{A}_\nu\star\mathcal{A}_\mu. \tag{3.30}$$

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That is, the operator $\partial_\mu \delta/\delta\sigma^{\mu\nu}(z)$ inserts the Maxwell equation in the noncommutative phase factor at the point z, as anticipated.

The standard loop equation of large-N Yang-Mills theory for the Wilson loop average $\langle W(C)\rangle$ emerges from eqs. (3.28) and (3.29) in the $q = n = 1$ case by putting $z = x$, summing over all x and using the formula

$$\frac{1}{N^2}\sum_x \Delta^{ij}(x)\Delta^{kl}(x) = N\,\delta^{il}\delta^{kj}\,, \qquad (q = n = 1 \text{ case})\,. \tag{3.31}$$

This equation is quadratic in $\langle W(C)\rangle$ due to large-N factorization of correlators. To better understand the consequences of eqs. (3.28) and (3.29) for $q, n \neq 1$, let us consider the case of $\theta' = 0$, whereby the continuum limit is rank one commutative gauge theory, previously known as Maxwell's theory. The phase factor (3.9) for a closed loop is now gauge invariant since the star-product becomes the ordinary product, so that $\mathcal{U}(x; C)$ becomes the usual phase factor of electrodynamics which is independent of x while the g's cancel on the right-hand side of eq. (3.6). The loop equation for the average of the phase factor can be obtained by combining the averages of eqs. (3.28) and (3.29) which results in the standard linear loop equation

$$\partial_\mu \frac{\delta}{\delta\sigma^{\mu\nu}(z)}\Big\langle \mathcal{U}(C)\Big\rangle = \frac{1}{\beta\epsilon^4}\int \mathrm{d}\xi^\nu\ \delta(\xi - z)\Big\langle \mathcal{U}(C)\Big\rangle\,. \tag{3.32}$$

We have just illustrated by this simple example how the phase factors (3.9) can indeed correspond to observables in noncommutative gauge theories associated with the unitary matrix model. This is precisely the novel feature of the present matrix model that was pointed out in section 2, namely that in the large N limit it is possible to arrive at a U(1) continuum gauge theory.

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A. Proof of completeness

We will demonstrate that the $\Delta(x)$ defined by eq. (2.16) form a complete set for the space of solutions to the constraints (2.8), i.e. that any $N \times N$ complex matrix A that commutes with Ω_μ ($\mu = 1, 2$) can be written uniquely as

$$A = \frac{1}{(mq)^2}\sum_x A(x)\Delta(x)\,. \tag{A.1}$$

We first note that

$$\mathcal{E} \stackrel{\rm def}{=} \left\{ A \in {\rm gl}(N, \mathbb{C}) \;\middle|\; A\Omega_\mu = \Omega_\mu A\,, \quad \mu = 1, 2 \right\} \tag{A.2}$$

defines a linear subspace of $\mathrm{gl}(N, \mathbb{C})$ which has an inner product defined by $\mathrm{tr}(A^\dagger B)$ for $A, B \in \mathrm{gl}(N, \mathbb{C})$. The $\Delta(x) \in \mathcal{E}$ form an orthogonal set,

$$\frac{1}{N}\, \mathrm{tr}(\Delta(x)\Delta(y)) = (mq)^2 \delta_{x,y}\,. \tag{A.3}$$

We now consider the linear subspace $\mathcal{E}'$ of $\mathcal{E}$ spanned by $\Delta(x)$. We wish to show that $\mathcal{E}' = \mathcal{E}$. For this, we introduce a convenient orthogonal basis of $\mathrm{gl}(N, \mathbb{C})$. We first represent a generic element of $\mathrm{gl}(N, \mathbb{C})$ as

$$X = \sum_{p_1,p_2=0}^{mnq-1} \sum_{q_1,q_2=0}^{q-1} C_{p_1,p_2,q_1,q_2} (\Gamma_1)^{\dagger p_1} (\Gamma_2)^{p_2} \otimes (\tilde{\Gamma}_1)^{\dagger q_2} (\tilde{\Gamma}_2)^{q_1}\,. \tag{A.4}$$

Requiring that X commutes with Ω_μ, $\mu = 1, 2$, we find that the coefficients C_{p_1,p_2,q_1,q_2} in (A.4) vanish except when the conditions

$$\begin{aligned} p_1 - npq_1 &\in nq\mathbb{Z} \\ p_2 - nq_2 &\in nq\mathbb{Z} \end{aligned} \tag{A.5}$$

are satisfied. From (A.5) it follows that p_μ should be multiples of n, so we write them as $p_\mu = nm_\mu$, where m_μ are integers running from 1 to mq. Using the fact that p and q are co-prime, we find that (A.5) can be satisfied if and only if $q_1 = am_1 \pmod q$ and $q_2 = m_2 \pmod q$, where a is an integer satisfying (2.9). This means that a complete orthogonal basis of $\mathcal{E}$ can be given by J_{m_1,m_2} defined by (2.14), or by $\Delta(x)$ defined by (2.16), which is just an orthogonal transformation of J_{m_1,m_2}. Using the orthogonality (A.3) of the basis $\Delta(x)$, we can write the $A(x)$ in (A.1) as

$$A(x) = \frac{1}{N}\, \mathrm{tr}(A\, \Delta(x))\,. \tag{A.6}$$

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